Stability of the steady state of delay-coupled chaotic maps on complex networks

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(Received 16 January 2008; revised manuscript received 23 April 2008; published 20 May 2008)

We study the stability of the steady state of coupled chaotic maps with randomly distributed time delays evolving on a random network. An analysis method is developed based on the peculiar mathematical structure of the Jacobian of the steady state due to time-delayed coupling, which enables us to relate the stability of the steady state to the locations of the roots of a set of lower-order bound equations. For δ-distributed time delays (or fixed time delay), we find that the stability of the steady state is determined by the maximum modulus of the roots of a set of algebraic equations, where the only nontrivial coefficient in each equation is one of the eigenvalues of the normalized adjacency matrix of the underlying network. For general distributed time delays, we find a necessary condition for the stable steady state based on the maximum modulus of the roots of a bound equation. When the number of links is large, the nontrivial coefficients of the bound equation are just the probabilities of different time delays. Our study thus establishes the relationship between the stability of the steady state and the probability distribution of time delays, and provides a better way to investigate the influence of the distributed time delays in coupling on the global behavior of the systems.

DOI: 10.1103/PhysRevE.77.056212 PACS number(s): 05.45.Xt, 89.75.−k, 05.45.Ra, 05.65.+b

I. INTRODUCTION

The emergence of collective behavior is one of the most prominent characteristics of complex systems [1]. In many cases, such a complex system may be well modeled by an ensemble of local dynamical systems (linear or nonlinear) which are interconnected to form some kind of network. Recently, a lot of research activities have focused on the synchronization of coupled nonlinear systems evolving on complex networks [2–8]. It is believed in general that such studies can deepen our understanding of the mechanisms of various cooperative behaviors observed in real complex systems.

In principle, the complicated global behaviors of coupled systems on a network are determined by the topological structure of the underlying network, the networking strategies (e.g., weighting, coupling delays, etc.), as well as the local dynamics. In most real problems, the influences of these factors on global behavior are interwoven and difficult to isolate. In most studies, usually the influence of one facet is investigated in depth, while the others are kept as simple as possible. The emphasis generally is to explore the relationship between the topological structure of the underlying network and the global synchronous behaviors; for example, how the spectrum of the network is related to the synchronization behavior of the coupled systems [9]. The interactions between the coupled units are often taken to be instantaneous. From the communication point of view, taking instantaneous coupling assumes implicitly an infinitely large capability of information processing on each unit. This could be unrealistic in some situations, where the effect of limited capability of information processing, or finite communication times, is crucial. The time delays between interconnections then have to be accounted for in these situations [10].

The influence of the time-delayed interactions on the global behavior of coupled nonlinear systems evolving on a network has been investigated recently [7]. It was found numerically that, in a certain range of the coupling strength, with a local chaotic map on each node and the nodes coupled together according to the links of the underlying network but with randomly distributed time delays, the system will settle on a steady state. In this stable homogeneous state, all local systems are in the same state, which is an unstable fixed point of the local dynamics. Such a phenomenon appears to be unique to time-delayed coupled systems, and is closely related to the consensus problems in multiagent-multivehicle systems [12].

An important question arises naturally: Under what conditions will the time-delayed coupled system go into the steady state? To answer this question, it is necessary to establish the relationship between the stability of the steady state and the various system parameters. This paper is devoted to the stability analysis of the steady state. Our analysis reveals the connections between the stability of the steady state, the coupling strength, the distribution of delay times, and the characteristics of the underlying network, as will be shown later.

The method we developed for the stability analysis of the steady state is different from the usual way of directly evaluating the eigenvalues of the Jacobian. The key observation is that, in the augmented phase space (defined below), the local Jacobian of the steady state is in the form of a block companion matrix [13,14]. This peculiar mathematical structure makes it possible to relate the stability of the steady state to the locations of the latent roots (or eigenvalues) of a lower-order matrix polynomial (or λ matrix, which is in a similar form to an ordinary polynomial but with matrices as coefficients [14]) associated with the Jacobian. This provides a uniform approach to deal with both the case where time delays are all the same (δ distributed) and the more general case where the delay times follow some arbitrary distribution. In the former case, the diagonalization procedure is applicable and precise analytic results can be obtained. We find that the stability of the steady state is determined by the maximum modulus of the roots of a set of polynomials, whose nontrivial coefficient is one of the eigenvalues of the weighted adjacency matrix of the underlying network. The same idea can be extended to deal with the general case. However, no simple conclusion can be obtained due to the fact that coefficients of the associated λ matrix are now randomly distributed. We find a necessary condition to bound the allowed coupling strength to achieve the steady state, based on the maximum modulus of the roots of an algebraic bound equation. In particular, when the number of the links is large, this necessary condition is related to the root of the maximum module of a bound polynomial whose nontrivial coefficients are the probabilities of the delay times. This interesting result makes it possible to study the effects of different time-delay distributions through the bound polynomials. It can also be shown that the above condition determines the stability of the fixed point of a weighted mean-field variable, which provides an alternative interpretation of the proposed necessary condition.

In the remainder of the paper, we first study the simpler situation where the delay times are all the same (δ distributed). The more complicated situation where the delay times are distributed arbitrarily is then analyzed. Finally, concluding remarks are presented.

II. STABILITY ANALYSIS FOR δ-DISTRIBUTED TIME DELAYS

Consider a general network described by an adjacency matrix A. An ensemble of discrete time dynamical systems evolve on the network. The state x_i of the local system on node i is governed by the following system equations:

$$x_i(n+1) = f(x_i(n)) + \frac{\epsilon}{l_i}\sum_{j=1}^{N} a_{ij}[f(x_j(n-D_{ji})) - f(x_i(n))],$$

$$(i=1,2,\ldots,N), \tag{1}$$

where N is the total number of nodes, a_{ij} is the (i,j) element of adjacency matrix A, l_i is the degree of node i, serving as the normalization constant, D_{ji} is the time delay of the connection from node j to node i, and ϵ is the coupling strength. The instantaneous links are not considered in our model. As shown in previous studies [7,11], in a certain range of the coupling strength ϵ, the above system will settle on a stable steady state, which is defined as $x_i(n)=x_j(n)=x_s$, $\forall\ i,j$ and $n>n_t$, where n_t is the transition time and $x_s=f(x_s)$ is an unstable fixed point of the local dynamics. This kind of behavior is very different from that of the coupled system without time delays. In instantaneously coupled systems, the global steady state is unstable and cannot be observed.

The main purpose of our study is to explore the relationship between the stability of the steady state, the coupling strength ϵ, the control parameters of the local system, the distribution of the time delays, and the network characteristics. To better present our results, we first consider a particular case where the delay times are identical. The system Eq. (1) then becomes

$$x_i(n+1) = f(x_i(n)) + \frac{\epsilon}{l_i}\sum_{j=1}^{N} a_{ij}[f(x_j(n-D)) - f(x_i(n))], \tag{2}$$

where D is a constant time delay. Let $x(n) = [x_1(n),x_2(n),\ldots,x_N(n)]^T$ and $X(n)=[x(n),x(n-1),\ldots,x(n-D)]^T$. The $(D+1)N$-dimensional variable $X(n)$ defines an augmented phase space which consists of the current states and all the previous states up to D time steps. Let $I_{N\times N}$ be an $N\times N$ identity matrix, $O_{N\times M}$ a zero matrix with size $N\times M$, and L a diagonal matrix with $(l_1,l_2,\ldots,l_N)$ as the main diagonal entries; then Eq. (2) can be rewritten in matrix form as

$$X(n+1) = T(X(n)) + SX(n), \tag{3}$$

where

$$T = \begin{bmatrix} (1-\epsilon)f\circ I_{N\times N} & O_{N\times(D-1)N} & \epsilon L^{-1}Af\circ I_{N\times N} \\ O_{DN\times N} & O_{DN\times(D-1)N} & O_{DN\times N} \end{bmatrix}$$

and

$$S = \begin{bmatrix} O_{N\times(D-1)N} & O_{N\times N} \\ I_{(D-1)N\times(D-1)N} & O_{(D-1)N\times N} \end{bmatrix},$$

where f represents the local dynamical system, and $f \circ I_{N\times N}X$ means applying f to every element of $I_{N\times N}X$. The steady state of the system can be written as $X_s=x_s I_{(D+1)N\times 1}$, where $I_{(D+1)N\times 1}$ is a $(D+1)N$-dimensional vector and all of its elements are equal to 1.

The local stability of the steady state of the system (3) is determined by the eigenvalues of the Jacobian of the fixed point X_s. It is easy to see that

$$J = \begin{bmatrix} (1-\epsilon)f'(x_s)I_{N\times N} & O & \cdots & O & \epsilon f'(x_s)L^{-1}A \\ I_{N\times N} & O & \cdots & O & O \\ O & I_{N\times N} & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & I_{N\times N} & O \end{bmatrix}, \tag{4}$$

where O is an $N\times N$ zero matrix.One immediately recognizes that J is in the form of a block companion matrix, associated with the matrix polynomial (or λ matrix)

$$p(\lambda) = \lambda^{D+1}I_{N\times N} - \lambda^{D}(1-\epsilon)f'(x_s)I_{N\times N} - \epsilon f'(x_s)L^{-1}A.$$

The eigenvalues of J are closely related to $p(\lambda)$ [13]. Indeed, given a λ matrix in general form $M(\lambda)=\lambda^m+A_1\lambda^{m-1}+\cdots+A_m,(A_k:N\times N)$, a latent root (or an eigenvalue) of $M(\lambda)$ is defined as a scalar λ such that the λ matrix $M(\lambda)$ is singular. Associated with $M(\lambda)$, there is a block companion matrix, reading

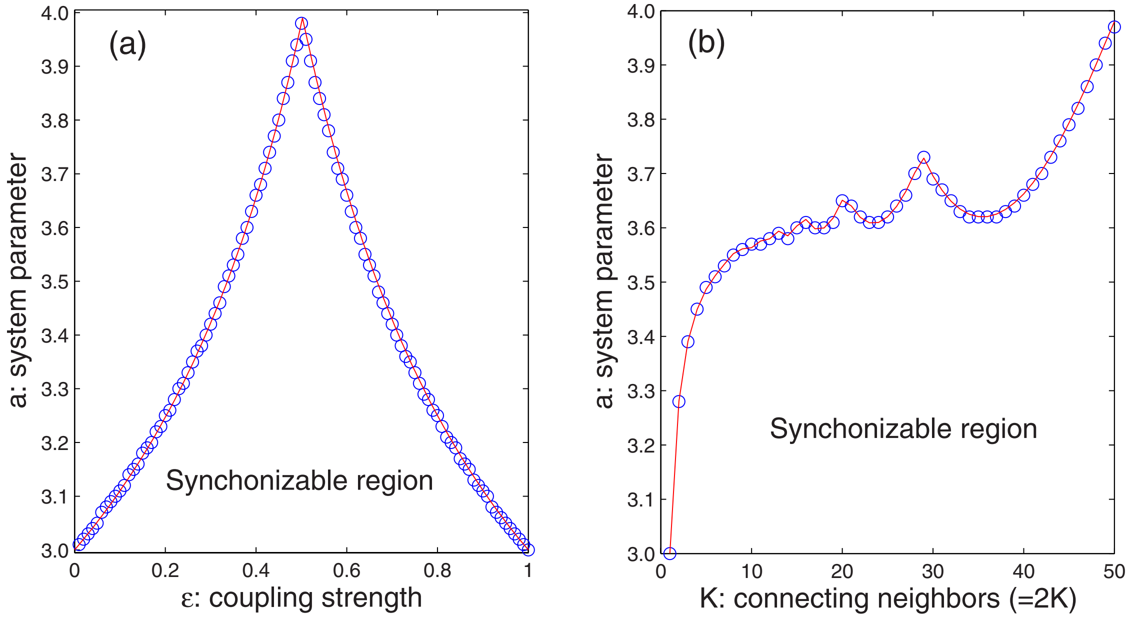

FIG. 1. (Color online) Relationship between the synchronizable system parameter a of the logistic map, the coupling strength ϵ, and the network parameter K. The relationship between the synchronizable system parameter a and the coupling strength ϵ is shown in (a) for a fully connected network with size $N=200$. (b) reveals the dependence of the synchronizable system parameter a on network parameter K. The network is K-nearest-neighbor connected (so the degree is $2K$ for each node) and the network size is $N=101$. In both figures, the circles are numerical results and the lines are theoretical predictions [Eq. (9) or Eq. (11) with corresponding Λ_N]. The synchronizable regions are indicated in the figures.

$$C=\begin{bmatrix} -A_1 & -A_2 & \cdots & -A_{m-1} & -A_m \\ I_{N\times N} & O & \cdots & O & O \\ O & I_{N\times N} & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & I_{N\times N} & O \end{bmatrix}.$$

Direct calculation shows that

$$\det(C-\lambda I_{mN\times mN}) = (-1)^{mN}\det(\lambda^m I_{N\times N} + A_1\lambda^{m-1} + \cdots + A_m).$$

Therefore, the eigenvalues of a block companion matrix C are the latent roots of the associated λ matrix $M(\lambda)$.

Such a relationship permits us to bring the theory of matrix polynomials to bear on the analysis of the eigenvalues of block companion matrices. To determine the eigenvalues of J, one simply sets $\det[p(\lambda)]=0$. In our model, this leads to

$$\det(\lambda^{D+1}I_{N\times N} - \lambda^D(1-\epsilon)f'(x_s)I_{N\times N} - \epsilon f'(x_s)L^{-1}A) = 0. \tag{5}$$

Let us define the normalized adjacency matrix $N = L^{-1/2}AL^{-1/2}$, which is symmetric and orthogonally diagonalizable. Let $S^{-1}NS=\Lambda$, where Λ is a diagonal matrix, and its diagonal entries $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_N$ are the eigenvalues of N. Since N and $L^{-1}A$ have the same eigenvalues, it is not difficult to derive that $1=\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_N \geq -1$. Multiplied by $S^{-1}L^{1/2}$ from the left and $L^{-1/2}S$ from the right, Eq. (5) becomes a set of ordinary algebraic equations

$$\lambda^{D+1} - (1-\epsilon)f'(x_s)\lambda^D - \epsilon f'(x_s)\Lambda_i = 0 \quad (i=1,2,\ldots,N). \tag{6}$$

Since the roots of these equations, λ_k, $k=1,2,\ldots,(D+1)N$, are just the eigenvalues of the Jacobian (4), they determine the local stability of the steady state of the system (2). Specifically, the steady state is stable if $|\lambda_k|<1$ for all k, and unstable otherwise. For this reason, Eqs. (6) are referred to as the bound equations in this paper.

To demonstrate the above procedure, we consider a simple situation where the time delay is $D=1$. Equations (6) become quadratic equations. For a quadratic equation in the general form $\lambda^2+a\lambda+b=0$, we have

$$|\lambda_{1,2}| \leq 1 \Rightarrow \begin{cases} b^2 < 1, \\ a^2 < (1+b)^2. \end{cases} \tag{7}$$

In our system, $a=-(1-\epsilon)f'(x_s)$ and $b=-\epsilon f'(x_s)\Lambda_i$. To make the global steady state stable, the coupling strength ϵ must satisfy

$$\epsilon^2|f'(x_s)|^2\Lambda_i^2 < 1 \quad \text{and} \quad (1-\epsilon)^2|f'(x_s)|^2 < [1-\epsilon f'(x_s)\Lambda_i]^2. \tag{8}$$

Equation (8) shows that the detailed structure of the local dynamics takes on a crucial role in the stability of the steady state and therefore the global collective behavior. More specifically, when $f'(x_s)>1$, condition (8) is inconsistent. Therefore, in this specific case, i.e., when the time delays are identical and $D=1$, the unstable fixed point with positive derivative cannot be the final steady state. When $f'(x_s) < -1$, condition (8) can be simplified as

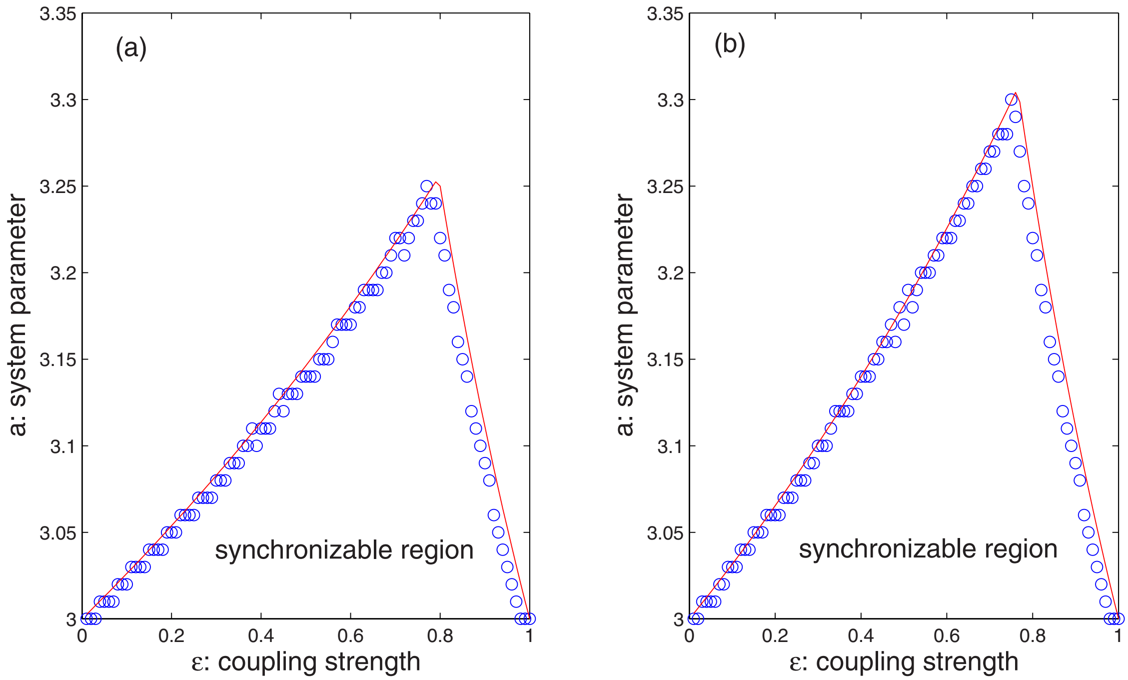

FIG. 2. (Color online) Relationship between the synchronizable system parameter a of the logistic map and the coupling strength ϵ for random networks. In (a) the network is an ER random network with size $N=100$, and average degree 5. In (b) the network is a scale-free network (BA model [16]) with size $N=100$. The fully connected initial nodes $d_0=4$ and attached links $m=3$. In both panels, the circles are numerical results and the lines are theoretical predictions [Eq. (9) or Eq. (11) with corresponding Λ_N]. The synchronizable regions are indicated in the figures.

$$\frac{d-1}{d(1+\Lambda_N)} \leq \epsilon \leq \frac{1}{d\Lambda_1} = \frac{1}{d}, \tag{9}$$

where $-1 \leq \Lambda_N \leq \Lambda_{N-1} \leq \cdots \leq \Lambda_2 \leq \Lambda_1 = 1$ are eigenvalues of the normalized adjacency matrix N, and $d=|f'(x_s)|>1$. In condition (9), Λ_N is determined by the topological structure of the network, and d is related to the fine details of the local dynamics. These two factors, together with the coupling strategy, thus determine the final global collective behavior.

To get concrete results, one needs to specify the local dynamics and the topology structure of the network. In the following, we adopt the logistic map as the local dynamical system, and apply the above method on two particular kinds of networks which permit analytic treatment. Let $f(x)=ax(1-x)$ $(3.0 \leq a \leq 4)$; then $x_s=1-1/a$ [as discussed above, another unstable fixed point $x_s'=0$ cannot be the stable steady state since $f'(0)=a>1$] and $d=|f'(x_s)|=a-2$. First, we consider the globally coupled network with $N=100$. In this situation, $L=1/(N-1)I_{N\times N}$, and $\Lambda_1=1$, $\Lambda_2=\Lambda_3=\cdots=\Lambda_N=-1/(N-1)$. The stability condition (9) becomes

$$\frac{(N-1)(a-3)}{(N-2)(a-2)} \leq \epsilon \leq \frac{1}{(a-2)}. \tag{10}$$

To verify the above theoretical results, we investigate numerically the stability of the steady state of time-delayed coupled logistic maps. The ranges of the parameters examined are $3.0 \leq a \leq 4.0$ and $0<\epsilon<1.0$ with step $\delta a=\delta\epsilon=0.01$. The coupled systems start to evolve, given random initial conditions, toward the steady state X_s. After some transient time $n_t=500$, the synchronization errors $e_s=\sum_{i=1}^{N}(x_i-x_s)^2$ are calculated. These are then quantized according to a small number 0.0001, i.e., $e=0$ if $e_s<0.0001$ and $e=1$ otherwise. Finally, the boundary between the stable and unstable ranges is extracted and compared with the theoretical values. The results of the numerical simulations are shown in Fig. 1(a). Not surprisingly, they match perfectly with the theoretical predictions.

One important consequence according to the conditions (9) and (10) is that the stability conditions may depend on Λ_N, for example,

$$a \leq \begin{cases} 2 + \dfrac{1}{1-\epsilon(1+\Lambda_N)} & \text{when} \quad 0 \leq \epsilon \leq \dfrac{1}{2+\Lambda_N}, \\ 2 + \dfrac{1}{\epsilon} & \text{when} \quad \dfrac{1}{2+\Lambda_N} \leq \epsilon \leq 1, \end{cases} \tag{11}$$

which can be related to the network size N and the network structure further. In a globally coupled network, because of condition (9), we have, regardless of the value of ϵ,

$$a \leq 4 - 1/(N-1), \tag{12}$$

where the equality is achieved only when $\epsilon=(N-1)/(2N-3)$. Although the synchronizable range of the parameter a increases with the size of the network, the system can never settle on the stable steady state when $a=4$. It can be seen clearly from Fig. 1(a) that, when a is close to 4, the coupled system cannot be synchronized to the desired steady state. For any coupling strength ϵ, the maximum permitted value of a can be predicted precisely using Eq. (11).

As another example that reveals the complicated relationship between the network structure and the synchronizable parameter a in achieving a stable steady state, consider a K-nearest-neighbor coupled network ($2K$ links per node) with $N=101$. In this case, $L=\frac{1}{2K}I_{N\times N}$. Let

$$c=[0,\underbrace{1,\ldots,1}_{K},\underbrace{0,\ldots,0}_{N-2K-1},\underbrace{1,\ldots,1}_{K}];$$

then the adjacency matrix A is a circulant matrix generated by c. The eigenvalues of the normalized adjacency matrix N are

$$\Lambda_i=\frac{1}{2K}\mathcal{F}(c)=\frac{1}{2K}\sum_{m=1}^{N}c_m e^{-j2\pi(m-1)(i-1)/N}\quad(1\le i\le N),$$

where $\mathcal{F}$ is the fast Fourier transform. Therefore, $\Lambda_1=1$, and

$$\Lambda_N=\underbrace{\min}_{m=2,3,\ldots,N}\frac{1}{2K}\left[\sin\left(\frac{2\pi(m-1)(2K+1)}{2N}\right)\Big/\sin\left(\frac{2\pi(m-1)}{2N}\right)-1\right].$$

For the case with the coupling strength $\epsilon=0.5$, we investigate the relationship between the system parameter a for achieving the stable steady state and the network structure parameter K. Similar to the case in Fig. 1(a), the boundary between the stable and unstable ranges is extracted based on the quantized synchronization errors. The boundary is plotted in Fig. 1(b) to compare with the theoretic prediction based on Eq. (11).

The above approach is also applicable to more complex random networks, although the analytic expression of Λ_N cannot be obtained in general. In Fig. 2, we show the simulation results and theoretic predictions when the underlying network is an Erdös-Renyi (ER) random network [15] [Fig. 2(a)] and a scale-free network [16] [Fig. 2(b)]. Again, the match is perfect.

For the situations where the delay times are the same but not equal to 1, the diagonalization procedure described above is also applicable. However, the stability conditions can be dramatically changed for different time delays. For example, when $3.0\le a\le 4.0$, if all time delays are equal to 2 in a globally coupled network, it can be shown that the maximum modulus of the roots of Eq. (6) ($i=1$) is always greater than 1. This implies that, when the system parameter a is in the above range, the steady state becomes unstable and cannot be observed when the time delays are changed from 1 to 2.

III. STABILITY ANALYSIS FOR GENERAL DISTRIBUTION OF TIME DELAYS

As shown above, in the situation where all time delays are the same, the adjacency matrix, together with a single time-delay value which could be regarded as one of the control parameter of the system, totally determines the interconnections among local systems. When the time delays are allowed to be distributed generally, the stability analysis of the global steady state becomes more difficult. In such a case, the topological structure of the underlying network, or equivalently the adjacency matrix alone, is inadequate to fully describe the interconnections among local dynamical systems. The necessary information comprises not only whether or not two nodes are connected, but also the time delay of this link if it exists. Mathematically, we need not only the adjacency matrix $A_{N\times N}$, but also a kind of decomposition A_k ($k=1,2,\ldots,D$) of A, such that each A_k is a (0,1) matrix with the same size as A, and $A_k(i,j)=1$ only if nodes i and j are connected by a link with time delay k. Therefore, A_k can be regarded as a reduced adjacency matrix describing an embedded subnetwork (or subgraph) which consists of all nodes but only the links where the time delay of the coupling are k. Obviously, the following relationship between A_k and A must be satisfied (note that no instantaneous links are allowed in our model):

$$\sum_{k=1}^{D}A_k=A, \tag{13}$$

where D is the maximum time delay. Furthermore, since the time delay of link $i\to j$ may be different from that of the link $j\to i$, the reduced adjacency matrix A_k may not be symmetric although the adjacency matrix A is. Unless the set A_k is given either deterministically or statistically, it is impossible to make any further analysis of the stability of the steady state.

Given A_k, the whole system is governed by an equation with the same form as Eq. (3), but with more complicated T,

$$T=\begin{bmatrix}(1-\epsilon)f\circ I_{N\times N} & \epsilon L^{-1}A_1 f\circ I_{N\times N} & \cdots & \epsilon L^{-1}A_{D_max}f\circ I_{N\times N}\\ O_{DN\times N} & O_{DN\times N} & \cdots & O_{DN\times N}\end{bmatrix}.$$

It is easy to verify that the Jacobian of the global steady state in this situation still has the form of a block companion matrix. The associated λ matrix, however, has to be modified to a more general form as follows:

$$p(\lambda)=\lambda^{D+1}I_{N\times N}-\lambda^{D}(1-\epsilon)f'(x_s)I_{N\times N}-\epsilon f'(x_s)L^{-1}\left(\sum_{k=1}^{D}\lambda^{D-k}A_k\right). \tag{14}$$

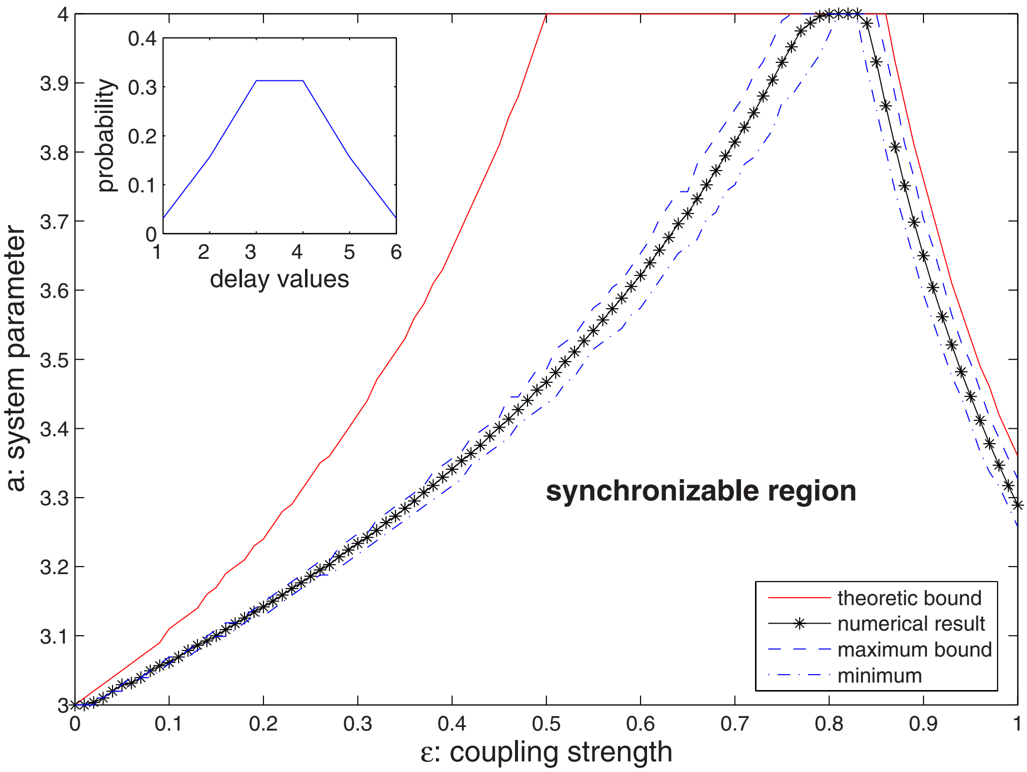

FIG. 3. (Color online) Bounds of the synchronizable region with different system parameters a of the logistic map and coupling strength ϵ. The size of the random network (ER model) is $N=100$, and the average degree is $d=30$. The time delays of the connections follow a binomial distribution (as shown in the inset). The synchronizable regions are indicated on the figures. The numerical results represent the average of $m=50$ different realizations. The maximum and minimum values of the bounds in 50 realizations for each coupling strength are also shown. The theoretical bound is calculated from Eq. (18) for each ϵ.

Finding the latent roots (or eigenvalues) of the above λ matrix is difficult generally. Only under a very special condition, i.e., when all $L^{-1}A_k$ can be diagonalized simultaneously by an orthogonal matrix, can the diagonalization procedure described above to deal with δ-distributed time delays be adopted to simplify the problem to finding the zeros of N ordinary polynomials of $(D+1)$ order. Obviously, these are extremely strong constraints to put on the structure of the network and the distribution of the time delays to apply the diagonalization procedure. Furthermore, when we are talking about certain random distributions of time delays of the connections, only the probability distribution is supposedly known. The information about the exact value of the time delay for each link is generally unknown.

In the following, we consider only the situations where each reduced adjacency matrix A_k is symmetric to simplify the problem. We are pursuing the modest goal of finding a suitable necessary condition (or a bound) of the parameters to achieve the stable steady state.

Suppose the probability distribution p_i, $i=1,2,\ldots,D$, of the time delays is known. First notice that λ matrix (14) has the same latent roots as

$$\bar{p}(\lambda)=\lambda^{D+1}I_{N\times N}-\lambda^{D}(1-\epsilon)f'(x_s)I_{N\times N}-\epsilon f'(x_s)\times\left(\sum_{k=1}^{D}\lambda^{D-k}N_k\right), \tag{15}$$

where $N_k=L^{-1/2}A_kL^{-1/2}$ is symmetric. (Note that N_k here is different from the normalized adjacency matrix defined in the last section.) Let λ_i be one of the latent roots of the λ matrix (15) and $q_{N\times 1}$ the corresponding nonzero latent vector. We have

$$\bar{p}(\lambda_i)q=0. \tag{16}$$

Assuming $\|q\|=1$ and left-multiplying the above equation by q', we obtain

$$\lambda_i^{D+1}-f'(x_s)(1-\epsilon)\lambda_i^{D}-\epsilon f'(x_s)\sum_{k=1}^{D}c_k\lambda_i^{D-k}=0, \tag{17}$$

where $c_k=q'N_kq$. Now since N_k are symmetric, due to the extremal properties of the eigenvalues, $u_N^{(k)}\le c_k\le u_1^{(k)}$, where $u_N^{(k)}\le u_{N-1}^{(k)}\le\cdots\le u_1^{(k)}$ are the eigenvalues of N_k.

Since the eigenvalues of the Jacobian of the global steady state are just the latent roots of the λ matrix (15), and each of the latent roots of the λ matrix (15) satisfies one of the equations in (17) (some different roots may satisfy the same equation), the stability of the global steady state is determined by the roots of the set of algebraic equations (17), the bound equations. Specifically, the steady state is locally stable if and only if $|\lambda_i|<1$, for all $i=1,2,\ldots,(D+1)N$. Unfortunately, the coefficients c_k are unknown, and we cannot solve the equations without actually calculating the eigenvalues of the Jacobian. On the other hand, if the modulus of some root(s) of a particular equation in (17) is greater than 1 for certain values of a and ϵ in parameter space, then the steady state is unstable under these parameters a and ϵ. Thus, the conditions that the modulus of each root of any one particular equation in (17) is less than 1 can serve as a necessary condition for the stable global steady state. Notice that the values c_k lie within the range $[u_N^{(k)},u_1^{(k)}]$ and are determined by N_k, which in turn depend on the probability distribution p_i. Without further information, it is reasonable to assume that c_k are randomly picked in the above range independently. In certain realizations, c_k can be close to certain eigenvalues $u_j^{(k)}$, $1\le j\le N$, of the N_k. Therefore, the equation in (17) with $c_k=u_1^{(k)}$ can be taken as the bound equation. The necessary condition of a stable steady state is that the moduli of all roots of this bound equation are less than 1.

In principle, any set of the eigenvalues of N_k can be selected to construct the bound equation. The advantage of using the largest eigenvalues of N_k is that they are closely related to the distribution of time delays. Consider a general random network with size N and total links M. Suppose p_k, $k=1,2,\ldots,D$, is the probability that the time delay of a link is k, where D is the maximum delay time. Then, when the number of links of the network is large, the nonzero elements in the ith row of matrix A_k approach l_ip_k, and each row sum of the matrix $L^{-1}A_k$ approaches p_k; therefore p_k is the maximum eigenvalue of $L^{-1}A_k$. Since the eigenvalues of N_k are the same as those of $L^{-1}A_k$, the maximum eigenvalue of N_k is just p_k. We thus obtain the desired bound algebraic equation

$$\lambda^{D+1}-f'(x_s)(1-\epsilon)\lambda_i^D-\epsilon f'(x_s)\sum_{k=1}^{D}p_k\lambda_i^{D-k}=0, \qquad (18)$$

where $\Sigma_{k=1}^{D}p_k=1$. For a stable steady state, the necessary condition is that the maximum modulus of the roots of Eq. (18) must be less than 1.

The necessary condition of the stable steady state derived above can also be understood from the mean-field approximation. First let us introduce a weighted mean-field variable $y(n)=\frac{1}{M}\Sigma_{k=1}^{N}l_kx_k(n)$, where $M=\Sigma_{k=1}^{N}l_i$ is the total number of links of the network, and according to Eqs. (1) we obtain

$$y(n+1)=G(y(n),y(n-1),\ldots,y(n-D))=\frac{1}{M}\sum_{i=1}^{N}l_if(x_i(n))+\frac{\epsilon}{M}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}[f(x_j(n-D_{ji}))-f(x_i(n))]. \qquad (19)$$

Then y has a fixed point $y_s=y(n+1)=y(n)=x_s$. Obviously, when the system is in the steady state, i.e., $x_i(n)=x_j(n)=x_s$, $\forall\ i,j$ and $n>n_t$, y is always at this fixed point. If the steady state is stable, y_s is always locally stable. On the other hand, if y_s is unstable, then the steady state must be unstable. Therefore, the stable condition of y_s is a necessary condition for the stable steady state $x_sI_{N\times1}$.

The solution of $y=y_s$ defines a manifold $\frac{1}{M}\Sigma_{k=1}^{N}l_kx_k(n)=x_s$ in N-dimensional phase space, and we are not able to examine the Jacobian of every point in this manifold directly. However, for y_s unstable, it is sufficient to examine a specific point, where $x_k=x_s$, $k=1,2,\ldots,N$. If the Jacobian of this point has an eigenvalue greater than 1, y_s is unstable.

Similar to the method applied in the last section, we define the augmented variable of y as $Y(n)=[Y_1(n),Y_2(n),\ldots,Y_{D+1}(n)]^T=[y(n),y(n-1),\ldots,y(n-D)]^T$. The fixed point of Y can be written as $Y_s=y_sI_{(D+1)\times1}$. Let

$$\bar{L}_{(D+1)N\times(D+1)N}=\begin{pmatrix} L & \cdots & \cdots \\ \cdots & L & \cdots \\ \cdots & \cdots & L\end{pmatrix}.$$

Equation (19) can be rewritten as

$$\bar{X}(n+1)=\frac{1}{M}\bar{L}X(n+1)=\frac{1}{M}\bar{L}T(X(n))+\frac{1}{M}\bar{L}SX(n).$$

Notice that $Y_k(n+1)=\Sigma_{j=N(k-1)+1}^{Nk}\bar{X}_j(n+1)$, $k=1,2,\cdots,D+1$, and we have

$$Y(n+1)=G(Y(n))+SY(n), \qquad (20)$$

where $S_{D\times D}$ is an identical matrix, $G(Y(n))=[g(Y(n),0,\ldots,0)]^T$,

$$g(Y(n))=\frac{(1-\epsilon)}{M}\sum_{i=1}^{N}l_if(x_i(n))+\sum_{k=1}^{D}\frac{\epsilon}{M}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}^{(k)}f(x_j(n-k)),$$

and $a_{ij}^{(k)}$ is the (i,j) element of the reduced adjacency matrix A_k. Now, since

$$\frac{dG}{dy}=\sum_{i=1}^{N}\frac{\partial G}{\partial x_i}\frac{\partial x_i}{\partial y}=\sum_{i=1}^{N}\frac{\partial G}{\partial x_i},$$

we get the Jacobian of Y_s,

$$J=\begin{bmatrix} (1-\epsilon)f'(x_s) & \frac{\epsilon f'(x_s)}{M}\Sigma_{i,j=1}^{N}a_{ij}^{(1)} & \cdots & \frac{\epsilon f'(x_s)}{M}\Sigma_{i,j=1}^{N}a_{ij}^{(D-1)} & \frac{\epsilon f'(x_s)}{M}\Sigma_{i,j=1}^{N}a_{ij}^{(D)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

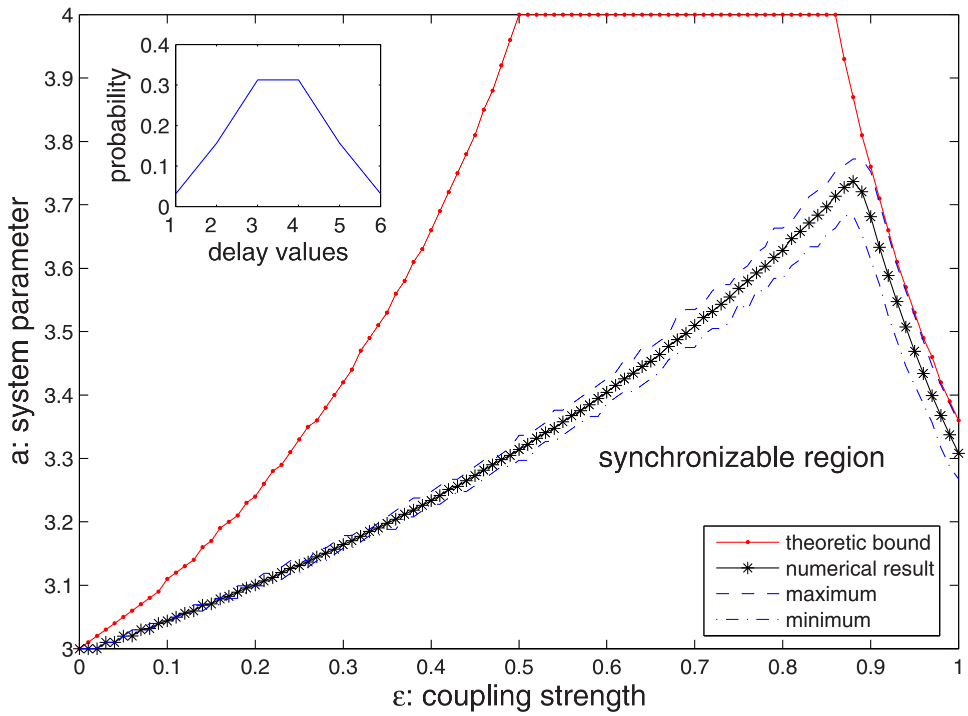

FIG. 4. (Color online) Bounds of the synchronizable region with different system parameters a of the logistic map and coupling strength ϵ. The size of the scale-free network (BA model) is $N=200$. The initial fully connected seeds $d_0=8$, and the attached links $m=7$. The time delays of the connections follow a binomial distribution (as in Fig. 3, and shown in the inset). The synchronizable regions are indicated in the figures. The numerical results represent the average of $m=50$ different realizations. The maximum and minimum values of the bounds in 50 realizations for each coupling strength are also shown. The theoretical bound is calculated from Eq. (18) for each ϵ.

The characteristic polynomial of J is

$$p_y(\lambda) = \lambda^{D+1} - (1-\epsilon)f'(x_s)\lambda^D - \frac{\epsilon f'(x_s)}{M}\sum_{k=1}^{D}\sum_{i,j=1}^{N} a_{ij}^{(k)}\lambda^{D-k},$$

and the eigenvalues of the Jacobian are the roots of the equation $p(\lambda)=0$. When the number of the links is large, or more specifically when the number of the nonzero elements in each row is close to its mean value, $\frac{1}{M}\Sigma_{i,j=1}^{N}a_{ij}^{(k)} \approx p_k$. This leads us back to the bound algebraic equation (18).

We examine the bound based on the maximum modulus of the roots of the above bound equation for pure random networks with large average degree and a scale-free network with smaller average degree. The simulation results are shown in Figs. 3 and 4. In both situations, the distribution of the time delays is binomial, $p_{k+1}=B(k,D-1,p)=\binom{D-1}{k}p^k(1-p)^{D-1-k}$, where $p=0.5$. For networks with different topologies or different distributions of delay times, the results are similar. In general, given a network with fixed links, the theoretical bound becomes tighter with more uneven distribution of delay times and larger maximum degree of the network. In this sense, for a network with a certain topological structure, inhomogeneity of time delays makes consensus easier to achieve. The necessary condition derived above can also be applied to the system where time delays are δ distributed. In such a case, the bound equation is just one of the equations in (6). Therefore, it is a necessary condition regardless of the number of links.

The proposed bound equation underscores the influence of the distribution of time delays on the stability property of the steady state. It provides a feasible way to study the effects of different time-delay distributions by investigating the locations of the roots of the associated bound equations.

IV. SUMMARY AND CONCLUSIONS

In summary, the stability of the global steady state of coupled nonlinear systems on random network with distributed time delays is investigated analytically and numerically in this paper. By constructing the augmented phase space, it is possible to relate the eigenvalues of the local Jacobian of the steady state to the latent roots (or eigenvalues) of the associated λ matrix. We find that in the situation where all time delays are identical, or δ distributed, the stability of the global steady state is determined by the maximum modulus of the roots of a set of algebraic equations, where the only nontrivial coefficient in each equation is an eigenvalue of the normalized adjacency matrix. In more general situations where the time delays are allowed to be distributed arbitrarily, we find a necessary condition for the stable steady state based on the maximum modulus of the roots of a bound equation, where the nontrivial coefficients are just the probabilities of time delays. It turns out that this condition is also the necessary condition that a weighted mean-field variable has a stable fixed point. Our study here establishes the relationship between the stability of the steady state and the probability distribution of the time delays, and provides a better way to investigate the influence of the distributed time delays in coupling on the global behavior of the system. We believe that the method developed in this paper also provides a useful tool to study the various problems where time-delayed couplings are involved.

ACKNOWLEDGMENTS

This work was supported by Temasek Laboratories at National University of Singapore through the DSTA Project No. POD0613356.

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